

ON NECKLACES INSIDE THIN SUBSETS OF \mathbb{R}^d

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ABSTRACT. We study similarity classes of point configurations in \mathbb{R}^d . Given a finite collection of points, a well-known question is: How high does the Hausdorff dimension $\dim_{\mathcal{H}}(E)$ of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, need to be to ensure that E contains some similar copy of this configuration? We prove results for a related problem, showing that for $\dim_{\mathcal{H}}(D)$ sufficiently large, E must contain many point configurations that we call k -necklaces of constant gap, generalizing equilateral triangles and rhombuses in higher dimensions. Our results extend and complement those in [3, 1], where related questions were recently studied.

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1. INTRODUCTION

The study of finite point configurations in sets of various sizes spans analysis, ergodic theory, number theory and combinatorics. A corollary (due to Steinhaus) of the Lebesgue density theorem states that any measurable set in \mathbb{R}^d with positive Lebesgue measure contains a similar copy of any finite configuration of points. There are many variations on this result. For instance, instead of sets of positive Lebesgue

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measure, one can consider an unbounded set $E \subset \mathbb{R}^d$ of positive upper Lebesgue density, in the sense that

$$\limsup_{R \rightarrow \infty} \frac{|E \cap [-R, R]^d|}{(2R)^d} > 0.$$

Here $|\cdot|$ denotes the d -dimensional Lebesgue measure. A result of Bourgain [2] (also Furstenberg, Katznelson and Weiss [6]) proves that E contains all sufficiently large copies of a non-degenerate k -simplex, i.e., a $(k+1)$ -point configuration, for $k \leq d$. Ziegler [18] has generalized this result for $k \geq d$, but the sufficiently large copies of the configuration are shown to be contained in an arbitrarily small neighborhood of E rather than in E itself. In particular, results of this type show that we can recover every simplex similarity type inside a subset of \mathbb{R}^d that is “large”, either in the sense of positive Lebesgue measure or of positive upper Lebesgue density. It is reasonable to wonder whether similar conclusions continue to hold even if such largeness assumptions are weakened. However, the following result due to Maga [10] shows that the conclusion in general fails for Lebesgue null sets in \mathbb{R}^d , *even if the set under consideration is of full Hausdorff dimension*. Let $\dim_{\mathcal{H}}(E)$ denote the Hausdorff dimension of a set $E \subset \mathbb{R}^d$.

Theorem 1.1. (Maga [10]) *The following conclusions hold.*

- (a) *For any $d \geq 2$, there exists a compact set $A \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(A) = d$ such that A does not contain the vertices of any parallelogram.*
- (b) *If $d = 2$, then given any nondegenerate triple of points x^1, x^2, x^3 in \mathbb{R}^2 , there exists a compact set $A \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(A) = 2$ such that A does not contain the vertices of any triangle similar to $\triangle x^1 x^2 x^3$.*

In view of Maga’s result, it is reasonable to ask whether interesting specific point configurations can be found inside thin sets under additional structural hypotheses. This question has been recently addressed by Chan, Laba and Pramanik [3], where the authors establish the existence of certain finite point configurations in sets of sufficiently high Hausdorff dimension and carrying a Borel measure with decaying Fourier transform. (The measure should also satisfy certain size bounds for Euclidean balls.) The point configurations obtained in [3] were required to obey appropriate nondegeneracy constraints when expressed as a linear system, and included both geometric and algebraic patterns such as corners in the plane, as well as polynomial-type configurations in \mathbb{R}^d . However, some natural configurations do not satisfy the non-degeneracy assumption of [3]. For example, *corners* in \mathbb{R}^3 , defined as collections of 4 points x, y, z, w in \mathbb{R}^3 such that

$$(1.1) \quad \begin{aligned} (x-y) \perp (x-z), \quad (x-y) \perp (x-w), \quad (x-z) \perp (x-w), \\ |x-y| = |x-z| = |x-w| \end{aligned}$$

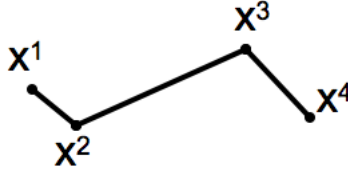


FIGURE 1. A 3-chain

are not covered by the setup of [3]. Neither does a *nonplanar* (i.e., not necessarily planar) *rhombus* in \mathbb{R}^3 , defined as a set of 4 points x, y, z, w such that $|x - y| = |y - z| = |z - w| = |w - x|$.

It is reasonable to ask which point configurations can be recovered without extra assumption on the Fourier decay. In view of Maga's result (Thm. 1.1 above), one cannot hope to prove nontrivial results of this type for configurations that contain a planar loop. However it still seems plausible that we may be able to handle tree-like point configurations and loops that are not contained in a plane and hence enjoy greater directional freedom. This question is partially addressed in [1]. To present this result, we need the following definition.

Definition 1.2. A k -chain in $E \subset \mathbb{R}^d$ with gaps $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \dots, x^{k+1} : x^j \in E, |x^{i+1} - x^i| = t_i > 0, 1 \leq i \leq k\}.$$

The k -chain has *constant gap* $t > 0$ if all the $t_i = t$. Finally, we say that the chain is *non-degenerate* if all the x^i 's are distinct.

See Fig. 1 for a depiction of a 3-chain.

Theorem 1.3. (Bennett, Iosevich and Taylor [1]) Suppose that $E \subset \mathbb{R}^d$ is a compact set, $d \geq 2$, and that $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. Then for any $k \geq 1$, there exists an open interval $I \subset \mathbb{R}$, such that for each $t \in I$ there exists a non-degenerate k -chain in E with constant gap t .

The idea behind the proof of Thm. 1.3 is to construct a measure on all k -chains, naturally induced from a Frostman measure μ on E , and consider its Radon-Nikodym derivative. We prove that it is bounded from above in all cases, and from below in the case when all the gaps are in a suitable interval. The lower bound is accomplished using the continuity of the distance measure in appropriate dimensional regimes. An upper bound is proved using a fractal variant of the classical Parseval identity recently established by Iosevich, Sawyer, Taylor and Uriarte-Tuero [9], based on an earlier result of Strichartz [16]. In practice, this amounts to obtaining upper and

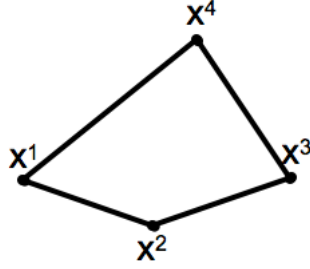


FIGURE 2. A 4-necklace

lower bounds on the quantity

$$(1.2) \quad C_k^\epsilon(\mu) = \int \dots \int \prod_{j=1}^k \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j)$$

that are uniform in ϵ . Here and throughout the paper, σ_t is the Lebesgue measure on the sphere of radius t , $\sigma_t^\epsilon = \sigma_t * \rho_\epsilon$, with $\rho \geq 0$ a smooth cut-off function, $\int \rho(x) = 1$ and $\rho_\epsilon(x) = \epsilon^{-d} \rho(\frac{x}{\epsilon})$. An analogous multilinear form, expressed in terms of the Fourier transforms of measures rather than the measures themselves, was used in [3] as well. There, a finite upper bound on the form justified its existence and definition; a nontrivial lower bound then established the existence of the linear configurations.

While the results in [1, 3] are focused on point configurations that do not contain loops, we shall see that both the lower bound and the upper bound idea in [1], combined with the generalized three-lines lemma approach in [9], allow us to capture configurations that were inaccessible by these previous methods. In particular, we will be able to handle nonplanar rhombuses in dimensions three and higher, as well as more complicated closed loops. We now turn our attention to the precise formulation of our results.

2. STATEMENT OF RESULTS

Definition 2.1. A k -necklace in $E \subset \mathbb{R}^d$, $d \geq 2$, with gaps $\tilde{\mathbf{t}} = (t_1, t_2, \dots, t_k)$, $t_j > 0$, is a finite sequence x^1, x^2, \dots, x^k , $x^j \in E$, such that $|x^j - x^{j+1}| = t_j$, $1 \leq j \leq k-1$ and $|x^k - x^1| = t_k$. We say that this necklace is *non-degenerate* if $x^i \neq x^j$ for any $1 \leq j \leq k$, and *has constant gap* t if $t_1 = \dots = t_k = t$.

Remark 2.2. Thus, a k -necklace is a closed $(k+1)$ -chain (see Fig. 2.), and being of constant gap is the same as all edges being of equal length.

Remark 2.3. A nondegenerate 4-necklace of constant gap $t > 0$ in \mathbb{R}^d is a nonplanar rhombus of side length t . Note that two non-degenerate 4-necklaces, even with similar

gap vectors, need not be similar to each other, due to the freedom that comes from not necessarily being planar.

Remark 2.4. In general a k -necklace with a given gap vector $\tilde{\mathbf{t}}$ is a member of the union of the similarity classes of a family of k -simplices, rather than being a similar copy of a specific k -simplex.

We now can state our main result.

Theorem 2.5. *Let E be a compact subset of \mathbb{R}^d , $d \geq 3$.*

i) Suppose that $d \geq 4$, k is even and $\dim_{\mathcal{H}}(E) > \frac{d+3}{2}$, without any additional assumptions on measures carried by E . Then there exists a non-empty open interval I such that for every $t \in I$, E contains some k -necklace with constant gap t .

ii) Suppose that $d \geq 3$. Suppose that for some $\delta > 0$, $\dim_{\mathcal{H}}(E) > d - \delta$ and there exists a Borel measure μ supported on E such that

$$(2.1) \quad |\hat{\mu}(\xi)| \leq C|\xi|^{-1-\frac{\delta}{2}}, \quad \forall \xi \in \mathbb{R}^d.$$

Then there exists a non-empty open interval I such that for every $t \in I$, E contains a nonplanar rhombus of side length t .

Remark 2.6. It would be interesting to extend Thm. 2.5 to cover the case when k is odd. Note however that, at least in the case $k = 3$, the conclusion of part (i) of the theorem is certainly false in view of Maga's counter-example [10].

Remark 2.7. If the $\dim_{\mathcal{H}}(E) = s$, then in (2.1), $1 + \frac{\delta}{2} \leq \frac{s}{2}$. In particular, if E is a Salem set [11] of dimension $s > \frac{d+2}{2}$, then E contains the vertices of a rhombus.

Remark 2.8. While we state Thm. 2.5 for necklaces with constant gaps, a careful examination of the proof shows that we can say a bit more:

Definition 2.9. We say that a non-degenerate $(n-1)$ -chain with vertices x^1, x^2, \dots, x^n generates a non-degenerate $(2n-2)$ -necklace with vertices $x^1, x^2, \dots, x^{2n-2}$ if $|x^j - x^{j+1}| = |x^{k+2-j} - x^{k+1-j}|$ for $2 \leq j \leq n-1$. (See Fig. 3.)

The proof of Thm. 2.5 (i) shows that in fact the conclusion holds for any necklace with an even number of vertices which is generated by a non-degenerate chain.

3. PROOF OF THEOREM 2.5

3.1. Preliminary calculations. We shall need the following result from [9], which we state in the form needed in this paper.

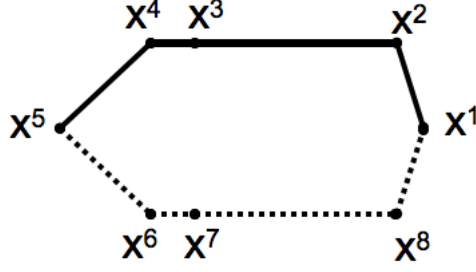


FIGURE 3. A 4-chain x^1, \dots, x^5 generates an 8-necklace x^1, \dots, x^8 .

Theorem 3.1. ([9, Thm 1.1]) *Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution satisfying*

$$|\widehat{K}(\xi)| \leq C|\xi|^{-\gamma}, \quad \gamma \in \left(0, \frac{d}{2}\right).$$

*For $\epsilon > 0$, let $K^\epsilon = K * \rho_\epsilon$. Suppose that ϕ, ψ are compactly supported Borel measures on \mathbb{R}^d satisfying $\phi(B(x, r)) \leq Cr^{s_\phi}$, $\psi(B(x, r)) \leq Cr^{s_\psi}$, respectively, with $s_\phi, s_\psi > 0$. Let $T_{K^\epsilon}f = K^\epsilon * (f\phi)$. Suppose that $\gamma > d - s$, where $s = \frac{s_\phi + s_\psi}{2}$. Then*

$$\|T_{K^\epsilon}f\|_{L^2(\psi)} \leq C\|f\|_{L^2(\phi)}$$

where C does not depend on ϵ .

Proof. Since the proof of Theorem 3.1 is very simple, we include it for the sake of completeness. It is enough to show that

$$\langle T_{K^\epsilon}f, g\psi \rangle \leq C\|f\|_{L^2(\phi)} \cdot \|g\|_{L^2(\psi)}, \quad \forall f, g.$$

The left hand side equals

$$\int \widehat{K^\epsilon}(\xi) \widehat{f\phi}(\xi) \widehat{g\psi}(\xi) d\xi.$$

By the assumptions of Theorem 3.1, the modulus of this quantity is bounded by

$$C \int |\xi|^{-\gamma} |\widehat{f\phi}(\xi)| |\widehat{g\psi}(\xi)| d\xi,$$

and applying the Cauchy-Schwarz inequality results in the following upper bound for this quantity:

$$(3.1) \quad C \left(\int |\widehat{f\phi}(\xi)|^2 |\xi|^{-\gamma_\phi} d\xi \right)^{\frac{1}{2}} \cdot \left(\int |\widehat{g\psi}(\xi)|^2 |\xi|^{-\gamma_\psi} d\xi \right)^{\frac{1}{2}}$$

for any $\gamma_\phi, \gamma_\psi > 0$ such that $\gamma = \frac{\gamma_\phi + \gamma_\psi}{2}$. By Lemma 3.4 soon to be proved below, the quantity (3.1) is bounded by $C\|f\|_{L^2(\phi)} \cdot \|g\|_{L^2(\psi)}$ after choosing, as we may, $\gamma_\phi > d - s_\phi$ and $\gamma_\psi > d - s_\psi$. This completes the proof of Theorem 3.1. \square

Let μ be a Frostman measure supported on E . Recursively define

$$(3.2) \quad d\mu_0(x) := d\mu(x); \quad d\mu_{k+1}^\epsilon(x) := \sigma_t^\epsilon * \mu_k^\epsilon(x) d\mu(x) =: f_{k+1}(x) d\mu(x), \quad k \geq 0.$$

Lemma 3.2. *Let $E \subset \mathbb{R}^d$ be compact with $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, and suppose that μ is a Frostman measure on E . If $d - \dim_{\mathcal{H}}(E) < \alpha < d$, then, with the notation in (3.2),*

$$(3.3) \quad \int |\widehat{\mu_k^\epsilon}(\xi)|^2 |\xi|^{-\alpha} d\xi \leq C(k) < \infty,$$

where $C(k)$ is independent of ϵ .

Remark 3.3. A careful examination of the proof shows that $C(k)$ above depends on the $(d - \alpha)$ -energy of μ , namely $\int |\widehat{\mu}(\xi)|^2 |\xi|^{-\alpha} d\xi$.

Proof. This lemma is proved in [1], but we give a proof for the sake of completeness. Begin by using (3.2) to rewrite the left hand side of (3.3) in the form

$$(3.4) \quad \int |\widehat{f_k \mu}(\xi)|^2 |\xi|^{-\alpha} d\xi.$$

Assuming Lemma 3.4 as stated below for the moment and applying it to (3.4), we see that

$$\int |\widehat{f_k \mu}(\xi)|^2 |\xi|^{-\alpha} d\xi \leq C \|f_k\|_{L^2(\mu)}^2.$$

We have thus reduced the issue to proving that $\|f_k\|_{L^2(\mu)}$ is bounded.

Define the operator $Tf(x) = \sigma_t^\epsilon * (f\mu)$. Observe that $f_k(x) = Tf_{k-1}(x)$. By Theorem 3.1,

$$(3.5) \quad \begin{aligned} \|f_k\|_{L^2(\mu)}^2 &= \|Tf_{k-1}\|_{L^2(\mu)}^2 \leq C \|f_{k-1}\|_{L^2(\mu)}^2 \leq C^k \|f_1\|_{L^2(\mu)}^2 \\ &\leq C^k \int (\sigma_t^\epsilon * \mu(x))^2 d\mu(x) \end{aligned}$$

and this quantity is bounded, once again, by Thm. 3.1. This completes the proof of Lemma 3.2, up to the proof of Lemma 3.4. \square

Lemma 3.4. *Let μ be a compactly supported Borel measure such that $\mu(B(x, r)) \leq Cr^s$ for some $s \in (0, d)$. Suppose that $\alpha > d - s$. Then for $f \in L^2(\mu)$,*

$$(3.6) \quad \int |\widehat{f\mu}(\xi)|^2 |\xi|^{-\alpha} d\xi \leq C' \|f\|_{L^2(\mu)}^2.$$

Proof. Let us observe that

$$(3.7) \quad \int |\widehat{f\mu}(\xi)|^2 |\xi|^{-\alpha} d\xi = C \int \int f(x) f(y) |x - y|^{-d+\alpha} d\mu(x) d\mu(y) = \langle Tf, f \rangle,$$

where

$$Tf(x) = \int |x - y|^{-d+\alpha} f(y) d\mu(y)$$

and the inner product above is with respect to $L^2(\mu)$. Observe that

$$\int |x - y|^{-d+\alpha} d\mu(y) \approx \sum_{j>0} 2^{j(d-\alpha)} \int_{|x-y|\approx 2^{-j}} d\mu(y) \leq C \sum_{j>0} 2^{j(d-\alpha-s)} \leq C'$$

since $\alpha > d - s$, where we have used $\text{diam}(\text{supp}(\mu)) < \infty$.

By symmetry, $\int |x - y|^{-d+\alpha} d\mu(x) \leq C'$ and Schur's test ([13], see also [15]) implies at once that

$$\|Tf\|_{L^2(\mu)} \leq C' \|f\|_{L^2(\mu)},$$

which implies that conclusion of Lemma 3.4 in view of (3.7) and the Cauchy-Schwarz inequality. The proof of Lemma 3.3 is thus complete. \square

We also need to show that the measure $d\mu_k^\epsilon$ is non-trivial. A variant of this result is at the core of the proof of the main result in [1], as explained in the paragraph following Thm. 1.3 above. See also [12] where it was originally shown that the set of distances determined by a set of Hausdorff dimension $> \frac{d+1}{2}$ contains an interval. For the background on the Falconer distance problem and the latest results see [5], [4] and [17].

Lemma 3.5. *With the notation above,*

$$(3.8) \quad \liminf_{\epsilon \rightarrow 0} \int d\mu_k^\epsilon(x) > 0,$$

provided that μ is a Frostman measure on a set of Hausdorff dimension $> \frac{d+1}{2}$.

Proof. To prove the lemma, assume inductively that

$$(3.9) \quad \liminf_{\epsilon \rightarrow 0} \int d\mu_{k-1}^\epsilon(x) > 0.$$

Note that this condition holds by definition if $k = 1$ due to the fact that μ is a probability measure supported on E . By (3.9) and Lemma 3.2,

$$(3.10) \quad \mu_{k-1} \equiv \lim_{\epsilon \rightarrow 0} \mu_{k-1}^\epsilon$$

is a non-zero Borel measure supported on E . This allows us to redefine $d\mu_k^\epsilon$ in (3.2) to equal

$$\sigma_t^\epsilon * \mu_{k-1}(x) d\mu(x).$$

We now write

$$\begin{aligned}
 (3.11) \quad \int d\mu_k^\epsilon(x) &= \int \sigma_t^\epsilon * \mu_{k-1}(x) d\mu(x) \\
 &= \int \widehat{\sigma}_t(\xi) \widehat{\rho}(\epsilon\xi) \widehat{\mu}_{k-1}(\xi) \widehat{\mu}(\xi) d\xi \\
 &= \int \widehat{\sigma}(t\xi) \widehat{\mu}_{k-1}(\xi) \widehat{\mu}(\xi) d\xi + R^\epsilon(t) \\
 &= M(t) + R^\epsilon(t).
 \end{aligned}$$

We now follow the argument in [8] to see that if $t > 0$, $M(t)$ is continuous and $\lim_{\epsilon \rightarrow 0} R^\epsilon(t) = 0$.

We have

$$M(t+h) - M(t) = \int (\widehat{\sigma}((t+h)\xi) - \widehat{\sigma}(t\xi)) \widehat{\mu}_{k-1}(\xi) \widehat{\mu}(\xi) d\xi.$$

The integrand goes to 0 as $h \rightarrow 0$, so we proceed using the dominated convergence theorem. If $t > 0$, the expression above is bounded by

$$\begin{aligned}
 &C(t) \int |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}_{k-1}(\xi)| |\widehat{\mu}(\xi)| d\xi \\
 &\leq C(t) \left(\int |\widehat{\mu}_{k-1}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}} \cdot \left(\int |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}}
 \end{aligned}$$

and this expression is finite by Lemma 3.2. We use the fact that, if $t \geq t_0 > 0$, the estimate $|\widehat{\sigma}(t\xi)| \leq C|\xi|^{-\frac{d-1}{2}}$ holds with C independent of t . This proves that $M(t)$ is continuous away from the origin.

We now prove that $\lim_{\epsilon \rightarrow 0} R^\epsilon(t) = 0$. We have

$$\begin{aligned}
 |R^\epsilon(t)| &\leq \int \widehat{\sigma}(t\xi) (1 - \widehat{\rho}(\epsilon\xi)) |\widehat{\mu}_{k-1}(\xi)| |\widehat{\mu}(\xi)| d\xi \\
 &\leq C \int_{|\xi| > \epsilon^{-1}} |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}_{k-1}(\xi)| |\widehat{\mu}(\xi)| d\xi \\
 &\leq C \left(\int_{|\xi| > \epsilon^{-1}} |\widehat{\mu}_{k-1}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{|\xi| > \epsilon^{-1}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}} \\
 &\leq C \left(\int_{\mathbb{R}^d} |\widehat{\mu}_{k-1}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{|\xi| > \epsilon^{-1}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}} \\
 &\leq C' \left(\int_{|\xi| > \epsilon^{-1}} |\widehat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \right)^{\frac{1}{2}},
 \end{aligned}$$

where in the last step we used Lemma 3.2 once again.

We conclude that it is enough to show that

$$(3.12) \quad \lim_{\epsilon \rightarrow 0} \int_{|\xi| > \epsilon^{-1}} |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}(\xi)|^2 d\xi = 0.$$

Using $s = \dim_{\mathcal{H}}(E) - \delta$ for arbitrarily small $\delta > 0$, we have

$$(3.13) \quad \lim_{\epsilon \rightarrow 0} \sum_{j > \log_2(\epsilon^{-1})} \int_{2^j \leq |\xi| \leq 2^{j+1}} |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}(\xi)|^2 d\xi.$$

Applying Lemma 3.6, to be proved below, we see that (3.13) is bounded by

$$\leq C \lim_{\epsilon \rightarrow 0} \sum_{j > \log_2(\epsilon^{-1})} 2^{-j \frac{d-1}{2}} \cdot 2^{j(d-s)}.$$

Hence, if $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$, the limit is 0. We have thus shown that

$$\lim_{\epsilon \rightarrow 0} \int d\mu_k^\epsilon(x) = M(t),$$

where $M(t) \geq 0$ is continuous function away from the origin. If we can show that $M(t)$ is not identically 0, it will follow that there exists an open interval I , on which, $M(t) > c > 0$. To see that $M(t)$ is not identically 0, rewrite (3.11) in the form

$$\int \int \sigma_t^\epsilon(x-y) d\mu(x) d\mu_{k-1}(y).$$

This quantity is comparable to the Radon-Nikodym derivative of the measure on $\Delta(E) = \{|x-y| : x, y \in E\}$ given by

$$\liminf_{\epsilon \rightarrow 0} \epsilon^{-1} \mu \times \mu_{k-1} \{(x, y) : t \leq |x-y| \leq t+\epsilon\}.$$

It follows that

$$\int M_k(t) dt = \int \int d\mu(x) d\mu_{k-1}(y)$$

and this quantity is strictly positive by (3.9) and the fact that μ is a probability measure. This proves that $M_k(t)$ is not identically 0 and thus completes the proof of Lemma 3.5. □

Lemma 3.6. *Suppose that μ is a compactly supported Borel probability measure on \mathbb{R}^d such that $\mu(B(x, r)) \leq Cr^s$. Then*

$$\int_{|\xi| \leq R} |\widehat{\mu}(\xi)|^2 d\xi \leq CR^{d-s}.$$

Proof. To prove the lemma, construct a smooth compactly supported function h such that

$$\int_{|\xi| \leq R} |\widehat{\mu}(\xi)|^2 d\xi \leq \int |\widehat{\mu}(\xi)|^2 \widehat{h}(\xi/R) d\xi.$$

This quantity is bounded by

$$R^d \int \int h(R(x-y)) d\mu(x) d\mu(y) \leq CR^{d-s},$$

as claimed. \square

3.2. Proof of Theorem 2.5 (i). Define

$$(3.14) \quad \mathcal{N}_k^\epsilon(\mu) = \int \dots \int \left\{ \prod_{j=1}^{k+1} \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j) \right\} \sigma_t^\epsilon(x^{k+1} - x^1) d\mu(x^{k+1}).$$

Since k is even, we may write $k = 2n - 2$ with n an integer. Observe that

$$(3.15) \quad \mathcal{N}_k^\epsilon(\mu) = \int \int \left\{ \int \dots \int \prod_{j=1}^n \sigma_t^\epsilon(x^{j+1} - x^j) \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^{n+1}).$$

3.2.1. Lower bound. Applying Cauchy-Schwarz to (3.15) we see that it suffices to obtain a lower bound for

$$(3.16) \quad \int \dots \int \left\{ \prod_{j=1}^n \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j) \right\} d\mu(x^{n+1}).$$

In other words, the Cauchy-Schwarz inequality turns a chain into a necklace. The case $k = 8$ is depicted in Fig. 3 above.

Observe that the quantity in (3.16) equals $\int d\mu_n^\epsilon(x)$ and we already proved in Lemma 3.5 above that the $\liminf_{\epsilon \rightarrow 0}$ of this quantity is positive.

3.2.2. Upper bound. Define $\mathcal{N}_k^{\epsilon, \alpha}$ by the formula

$$(3.17) \quad \mathcal{N}_k^{\epsilon, \alpha} = \int \int F(x^1, x^n)^2 d\mu(x^1) d\mu(x^n),$$

where

$$F(x^1, x^n) = \int \dots \int \sigma^{\epsilon, -\alpha}(x^2 - x^1) \sigma_t^{\epsilon, -\alpha}(x^n - x^{n-1}) \prod_{j=2}^{n-1} \sigma_t^{\epsilon, \alpha}(x^{j+1} - x^j) \prod_{j=2}^{n-1} d\mu(x^j), \text{ and}$$

$$\sigma^\alpha(x) = \frac{1}{\Gamma(\alpha)} (1 - |x|^2)_+^{\alpha-1}, \quad \sigma^{\epsilon, \alpha} = \sigma^\alpha * \rho_\epsilon,$$

and α is a complex number. Recall the well-known fact (see e.g. [15, 14]) that

$$(3.18) \quad |\widehat{\sigma}_t^\alpha(\xi)| \leq C |\xi|^{-\frac{d-1}{2} - \operatorname{Re}(\alpha)}.$$

First consider the case $\operatorname{Re}(\alpha) = 1$, $n \geq 3$. Let $\alpha = 1 - iu$. Then

$$(3.19) \quad |\mathcal{N}_k^\alpha(\mu)| \leq \int \int \left\{ \int \dots \int G(x^1, x^2, x^{n-1}, x^n) \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n)$$

$$\leq \int \int \left\{ \int \int G(x^1, x^2, x^{n-1}, x^n) d\mu(x^2) d\mu(x^{n-1}) \right\}^2 d\mu(x^1) d\mu(x^n), \text{ where}$$

$$G = G(x^1, x^2, x^{n-1}, x^n) = |\sigma_t^{\epsilon, -1+iu}(x^2 - x^1)| |\sigma_t^{\epsilon, -1+iu}(x^n - x^{n-1})|.$$

Observe that

$$|\sigma_t^\alpha(x)| = \frac{1}{|\Gamma(\alpha)|} (1 - |x|^2)_+^{\operatorname{Re}(\alpha)-1}$$

and we shall denote $|\sigma_t^{\epsilon, \alpha}(x)| =: \lambda^{\epsilon, \alpha}(x)$. In order to bound (3.19), it suffices to show that

$$\int (\lambda^{\epsilon, -1+iu} * \mu(x))^2 d\mu(x) \leq C(u)$$

if μ is Frostman measure on a set of Hausdorff dimension $> \frac{d+3}{2}$. Since

$$(3.20) \quad |\widehat{\lambda^{\epsilon, -1+iu}}(\xi)| \leq C(u) |\xi|^{-\frac{d-3}{2}}$$

by (3.18) and its proof, the claim follows from Theorem 3.1. One can check using Stirling's formula that $C(u)$ grows like $Ce^{C|u|}$.

We now consider the case $\operatorname{Re}(\alpha) = -1$, $n \geq 3$. Then

$$(3.21) \quad |\mathcal{N}_k^\alpha(\mu)| \leq \int \int \left\{ \int \dots \int \prod_{j=2}^{n-1} |\sigma_t^{\epsilon, -1+iu}(x^{j+1} - x^j)| \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n)$$

$$= \int \int \left\{ \int \dots \int \prod_{j=2}^{n-1} \lambda^{\epsilon, -1+iu}(x^{j+1} - x^j) \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n).$$

Let $g_1(x) = \lambda^{\epsilon, -1+iu} * \mu(x)$ and define inductively $g_j(x) = \lambda^{\epsilon, -1+iu} * (g_{j-1}\mu)(x)$. By inspection, the expression in (3.21) equals

$$(3.22) \quad \int \int |g_n(x^n)|^2 d\mu(x^n) d\mu(x^1) = \int |g_n(x^n)|^2 d\mu(x^n).$$

Let $Tg(x) = \lambda^{\epsilon, -1+iu} * g(x)$. Then the right hand side of (3.22) equals

$$\int |Tg_{n-1}(x)|^2 d\mu(x).$$

Applying Thm. 3.1 repeatedly, recalling (3.20) and that μ is a Frostman measure, we see that this expression is $\leq C(n) \|g_1\|_{L^2(\mu)}^2$, provided that

$$\dim_{\mathcal{H}}(E) > d - \frac{d-3}{2} = \frac{d+3}{2}.$$

Applying Thm. 3.1 one last time, we see that $\|g_1\|_{L^2(\mu)}$ is finite and the proof of the upper bound when $n \geq 3$ is completed by applying the following variant of the classical Hadamard three lines lemma due to Hirschman.

Lemma 3.7. [7] *If Φ is a continuous function on the strip S that is holomorphic in the interior of S and satisfies the bound*

$$\sup e^{-k|Im(z)|} \log |\Phi(z)| < \infty, z \in S$$

for some constant $k < \pi$, then

$$\log |\Phi(\theta)| \leq \frac{\sin(\pi\theta)}{2} \int_{-\infty}^{\infty} \frac{\log |\Phi(iy)|}{\cosh(\pi y) - \cosh(\pi\theta)} + \frac{\log |\Phi(1+iy)|}{\cosh(\pi y) + \cosh(\pi\theta)} dy$$

for all $\theta \in (0, 1)$.

The proof of Thm. 2.5 will be complete once we address the upper bound in the case $n = 2$ and prove that at least some of the k -necklaces obtained are non-degenerate.

3.3. Proof of Theorem 2.5 (i) for $n = 2$. Consider

$$(3.23) \quad \int \int \left\{ \int \sigma_t^{\epsilon, \alpha}(x-z) \sigma_t^{\epsilon, -\alpha}(y-z) d\mu(z) \right\}^2 d\mu(x) d\mu(y).$$

Suppose that $Re(\alpha) = 1$. Then this quantity is bounded by

$$(3.24) \quad \begin{aligned} & \int \int \left\{ \int |\sigma_t^{\epsilon, -\alpha}(y-z)| d\mu(z) \right\}^2 d\mu(x) d\mu(y) \\ &= \int \int \left\{ \int \frac{1}{|\Gamma(\alpha)|} (1 - |y-z|^2)_+^{-Re(\alpha)-1} d\mu(z) \right\}^2 d\mu(x) d\mu(y). \end{aligned}$$

This quantity is bounded above by the proof of the case $Re(\alpha) = 1$, $n \geq 3$ above. Thus we are done by Lemma 3.7 because we arrive at the exact same expression taking $Re(\alpha) = -1$ and reversing the roles of the variables. This takes care of the upper bound. The lower bound for a general n is proved above.

3.4. Proof of Theorem 2.5 (ii). Rewrite the expression in (3.24) above in the form

$$\int \int (\lambda^{\epsilon, -1+iu} * \mu(y))^2 d\mu(y) d\mu(x) = \int (\lambda^{\epsilon, -1+iu} * \mu(x))^2 d\mu(x).$$

Before we apply Thm. 3.1, we need a simple calculation. Treating K as a measure, observe that

$$\lambda^{\epsilon, -1+iu}(B(x, r)) \leq Cr^{d-2}.$$

The proof follows by a direct calculation. We now apply Thm. 3.1 with $K = \mu$, $\phi = \lambda^{\epsilon, -1+iu}$ and $\psi = \mu$. We shall assume that

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\gamma}$$

for some $\gamma > 0$. It follows that the $L^2(\phi) \rightarrow L^2(\psi)$ bound holds, with $f \equiv 1$ if

$$\gamma > d - \frac{d-2+s}{2} = \frac{d}{2} + 1 - \frac{s}{2}.$$

In particular, this means that if $s = d - \delta$, for some $\delta > 0$, then

$$\gamma > 1 + \frac{\delta}{2}.$$

It remains to prove that at least some of the necklaces obtained above are non-degenerate.

3.5. The non-degeneracy argument. Suppose, without loss of generality, that $|x^1 - x^{j_0}| \leq N\epsilon$ for some $j_0 \neq 1, k$ and that $|x^1 - x^j| > N\epsilon$ for all $j < j_0$. See Fig. 4.

Integrating in $d\mu(x^{j_0})$ and noting that $\sigma^\epsilon(x^{j_0} - x^{j_0+1}) \leq C\epsilon^{-1}$, $\sigma^\epsilon(x^{j_0} - x^{j_0-1}) \leq C\epsilon^{-1}$, we see that the expression in (3.14), with the additional restriction that two vertices are within $N\epsilon$ of each other, is bounded by

$$C \cdot k \cdot (N\epsilon)^s \cdot \epsilon^{-2} \cdot C_{k-2}^\epsilon(\mu) \leq C' k N^s \epsilon^{s-2},$$

where $C_k^\epsilon(\mu)$ is defined in (1.2), and the fact that $C_{k-2}^\epsilon(\mu) \leq C$, independently of ϵ is proved in [1] and also follows easily from the fact, proved in the course of proving Lemma 3.2 above that $\|f_k\|_{L^2(\mu)}$, with f_k defined in (3.2) is bounded by a finite constant depending only on k .

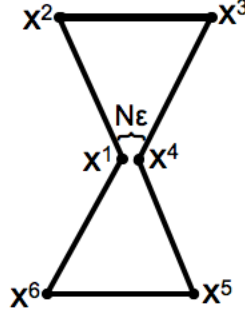


FIGURE 4. A bottleneck

We conclude that the integral

$$(3.25) \quad \int_S \left\{ \prod_{j=1}^{k-1} \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j) \right\} \sigma_t^\epsilon(x^k - x^1) d\mu(x^k), \text{ where}$$

$$S = \{(x^1, \dots, x^{k+1}) \in E^{k+1} : |x^1 - x^j| > N\epsilon; j \neq 1\},$$

is bounded from below by a non-zero constant as long as, say, $N < C\epsilon^{-1+\frac{2}{s}+\delta}$ for some $\delta > 0$. If $\delta > 0$ is chosen small enough, $\epsilon^{-1+\frac{2}{s}+\delta} \rightarrow \infty$ as $\epsilon \rightarrow 0$. Taking \liminf as $\epsilon \rightarrow 0$ we see that there exists a non-degenerate k -necklace with $\text{gap} \equiv t$.

4. CONCLUDING REMARKS

The purpose of this section is to put the methods of this paper into perspective and describe their limitations. In simple terms the approach of this paper can be described as follows. We use the Cauchy-Schwarz inequality to relate a chain to necklace with even number of vertices. This procedure allows us to obtain an immediate lower bound on the Radon-Nikodym derivative of the natural candidate for the measure on set of necklaces with prescribed gaps. We then obtain an upper bound on the Radon-Nikodym derivative using the three lines lemma and harmonic analytic inequalities, thus completing the proof of the assertion that vertices of the necklace can be found inside a compact subset of \mathbb{R}^d of a sufficiently large Hausdorff dimension.

The method of proof described above suggests that further progress may be possible if we use the results of this paper and then create more elaborate point configuration by the means of the Cauchy-Schwarz or Hölder's inequalities. What types of configuration can we hope to obtain in this way? In order to get the flavor, let's start with a 4-necklace and apply the Cauchy-Schwarz inequality in the x^1, x^2, x^3 -variables.

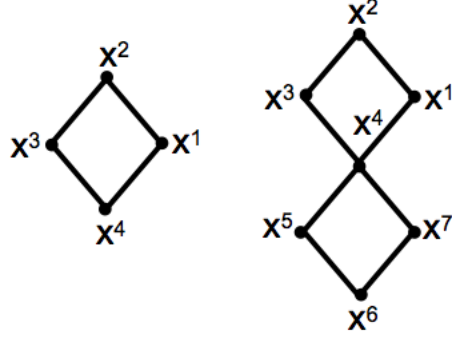


FIGURE 5. Cauchy-Schwarz turns a necklace into two necklaces sharing a point

We obtain

$$\begin{aligned}
 & \left[\int_{E^4} \prod_{j=1}^3 \sigma^\epsilon(x^j - x^{j+1}) \sigma^\epsilon(x^4 - x^1) \prod_{j=1}^4 d\mu(x^j) \right]^2 \\
 (4.1) \quad & \leq \int_E \left\{ \int_{E^3} \prod_{j=1}^3 \sigma^\epsilon(x^j - x^{j+1}) \sigma^\epsilon(x^4 - x^1) \prod_{j=1}^3 d\mu(x^j) \right\}^2 d\mu(x^4)
 \end{aligned}$$

$$\begin{aligned}
 (4.2) \quad & \leq \int_{E^7} \sigma^\epsilon(x^1 - x^4) \cdot \prod_{j=1}^3 \sigma^\epsilon(x^{j+1} - x^j) \times \\
 & \quad \sigma^\epsilon(x^4 - x^7) \cdot \prod_{j=4}^6 \sigma^\epsilon(x^{j+1} - x^j) \prod_{j=1}^7 d\mu(x^j),
 \end{aligned}$$

which is the Radon-Nikodym derivative of the natural measure on two 4-necklaces sharing the vertex x^4 . See Figure 5. Obtaining an upper bound for (4.2) is by no means trivial, but possible. We outline the argument because it leads to interesting harmonic analysis and illustrates the rich set of connections between geometric problems and harmonic analytic inequalities that these questions foster. Recalling the idea behind (3.23), we can express (4.1) in the form

$$(4.3) \quad \int \int \int F^2(x^2, x^4) G^2(x^7, x^4) d\mu(x^2) d\mu(x^4) d\mu(x^6),$$

where

$$F(x^2, x^4) = \int \sigma^\epsilon(x^2 - x^1) \sigma^\epsilon(x^4 - x^1) d\mu(x^1), \text{ and}$$

$$G(x^7, x^4) = \int \sigma^\epsilon(x^7 - x^4) \sigma^\epsilon(x^7 - x^6) d\mu(x^7).$$

Applying Cauchy-Schwarz yet again reduces matters to bounding the quantity

$$(4.4) \quad \int \int \left\{ \int \sigma^\epsilon(x^2 - x^1) \sigma^\epsilon(x^4 - x^1) d\mu(x^1) \right\}^4 d\mu(x^2) d\mu(x^4).$$

We pause for a moment to point out the difference between this quantity and (3.23), the expression we needed to bound to handle the rhombus (4-necklace). In (3.23) the inner expression in (4.4) is raised to the power of 2 instead of the power of 4. This naturally leads us to consider the L^4 version of Thm. 3.1, which can be obtained, with a worse yet still non-trivial lower bound on exponents s_ϕ and s_ψ , corresponding to the dimensional restriction, by a rather straightforward modification of the proof.

By the same method we can start with any necklace with an even number of vertices and by applying Hölder's inequality with the integer exponent $m \geq 2$ (positive integer), we obtain m necklaces sharing a common vertex. While it would be difficult to classify succinctly all the point configurations that can be obtained by starting with a chain and successively applying Hölder's inequality, this example is quite representative and also illustrates the limitations of our method.

There remain geometric configurations that cannot be handled either by the methods of this paper, or those in [3]. For example, the three-dimensional corner, described in (1.1) appears to be outside the reach of both methods. The authors hope to return to this issue in a sequel.

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